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Strong Berge and Pareto Equilibrium Existence for a Noncooperative Game

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Abstract

In this paper, we study the main properties of the strong Berge equilibrium which is also a Pareto efficient (SBPE) and the strong Nash equilibrium (SNE). We prove that any SBPE is also a SNE, we prove also existence theorem of SBPE based on the Ky Fan inequality. Finally, we also provide a method for computing SPBE.

Keywords : Strong Berge equilibrium, Pareto efficiency, strong Nash equilibrium, Ky Fan inequality

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1 Introduction

The Nash equilibrium (Nash [1950]) is probably the most important solution concept in game theory. It is immune to unilateral deviations, that is, each player has no incentive to deviate from his/her strategy given that the other players do not deviate. However, many games have several Nash equilibria which leads to a selection problem. Some authors proposed refinements which can be used to separate the reasonable from the unreasonable equilibria. Among these refinements, we can cite, for example, the perfect equilibrium (Selten [1975]), the proper equilibrium (Myerson [1978]), the sequential equilibrium (Kreps and Wilson [1982]). All these equilibria are related to one another in varying degrees. However, the ultimate refinement that exactly characterizes rational behavior can still include multiple equilibria for many games.

Berge [1957] introduced the strong Berge equilibrium. The strong Berge equilibrium is stable against deviation of all the players except one of them. Indeed, if a player chooses his strategy in a strong Berge equilibrium, then he obliges all the other player to do so. As a strong Berge equilibrium is also a Nash equilibrium, we can consider it as a refinement to this last one. The same authors showed a theorem of existence of this equilibrium. Aumann [1959] introduced the strong Nash equilibrium (SNE) which ensures a more restrictive stability than the Nash equilibrium. A SNE is a Nash equilibrium such that there is a nonempty set of players who could all gain by deviating together to some other combination of strategies which is jointly feasible for them, while all other players who are not in this set are expected to maintain their equilibrium strategies. Since this requirement applies to the grand coalition of all players, SNE must be Pareto efficient. Thus, a SNE is not only immune to unilateral deviations, but also to deviations by coalitions. We can then consider it as a refinement of the Nash equilibrium which is Pareto efficient.

The SNE has been used to study different noncooperative games as coalition formation (Hart and Kurz [1983], Hart and Kurz [1984], Bernhein *et al.* [1987], Chander and Tulkens [1997], Le Breton and Weber [2005]), congestion games (Hotzman and Law-Yone [1997], Voorneveld *et al.* [1999]), voting models (Keiding and Peleg [2001], Brams and Sanver [2006] and Moulin [1982]), network formation (Matsubayachi and Yamakawa [2006]), production externality games (Moulin and Shenker [1992] and Moulin [1994]), and many others economic situations: Abreu and Sen [1991], Tian [1999], Tian [2000], Suh [2003], Suh [1996]), Hirai *et al.* [2006], Konishi *et al.* [1997a], Konishi *et al.* [1997b], Konishi *et al.* [1997c], Nishihara [1999], Ray [2001], Yi [1999] and Young [1998]. This series of examples reveals the explanatory power of such equilibrium concept. However, these contributions are also paradoxical since there does not exist a general theorem which establishes clear existence conditions for the SNE.

In the present paper, we show that any strong Berge equilibrium which is Pareto efficient (SBPE) is also a strong Nash equilibrium. Furthermore, we propose necessary and sufficient conditions for the existence of both an SBPE and SNE. Our result is based on the Ky Fan inequality (Fan [1972]).

The paper is organized as follows. Section 2 introduces the strong Berge equilibrium, strong Nash equilibrium and some of their properties. In section 3, we introduce the concept of strong Berge equilibrium which is also Pareto efficient (SBPE), we show that an SBPE is also SNE and we establish necessary and sufficient conditions for the existence of an SBPE and provide a method for its computation. Section 4 concludes the paper.

2 Strong Berge and Nash Equilibria

In this section, we give the definitions of a strong Berge and Nash equilibria, its interpretations and some of its properties. Consider the following noncooperative game in normal form:

$$G = (X_i, f_i)_{i \in I} \quad (2.1)$$

where $I = \{1, \dots, n\}$ is the finite set of players, $X = \prod_{i \in I} X_i$ is the set of strategy profiles of the game, where X_i is the set of strategies of player i ; $X_i \subset E_i$, E_i is a vector space. $f = (f_1, f_2, \dots, f_n)$ where $f_i : X \rightarrow \mathbb{R}$ is the payoff function of player i .

The aim of each player in this game is to maximize his payoff function.

Let \mathfrak{S} denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $K \in \mathfrak{S}$, we denote by $-K$; the set $-K = \{i \in I \text{ such that } i \notin K\}$: the coalition of the players not in K , if K is reduced to a singleton $\{i\}$, we denote by $-i$ the set $-K$. We also denote by $X_K = \prod_{i \in K} X_i$ the set of strategies of the players in coalition K . If $\{K_i\}_{i \in \{1, \dots, s\} \subset \mathbb{N}}$ is a partition of the set of players I , then any strategy profile $x = (x_1, \dots, x_n) \in X$ can be written as $x = (x_{K_1}, x_{K_2}, \dots, x_{K_s})$ where $x_{K_i} \in \prod_{j \in K_i} X_j$.

Based on the concept of an equilibrium of a coalition R with respect to a coalition of Berge K Berge [1957], Abalo and Kostreva Abalo and Kostreva [1996b] introduced the following concept of equilibrium.

Definition 2.1 Consider a game (2.1). Let $R = \{R_t\}_{t \in M}$ be a partition of I and $S = \{S_t\}_{t \in M}$ be a set of subsets of I . A feasible strategy $\bar{x} \in X$ is an equilibrium point for the set R relative to the set S or simply a Berge equilibrium point for game (2.1) if

$$f_{r_m}(\bar{x}) \geq f_{r_m}(x_{S_m}, \bar{x}_{-S_m}),$$

for each given $m \in M$, any $r_m \in R_m$ and $x_{S_m} \in X_{S_m}$.

Abalo and Kostreva (Abalo and Kostreva [2005], Abalo and Kostreva [2004], Abalo and Kostreva [1996a] and Abalo and Kostreva [1996b]) provide an existence theorem of this equilibrium as Theorems 2, 3, 4, 5 and Corollary 3 (Abalo and Kostreva [2005]), Theorems 3.1 and 3.2 (Abalo and Kostreva [2004]), Theorems 2 and 3 (Abalo and Kostreva

[1996a]), and Theorems 3.2, 3.3, 3.4 and 3.5 (Abalo and Kostreva [1996b]). Nessah *et al.* [2007] found that the assumptions given in the Abalo and Kostreva's Theorem are not sufficient for the existence of Berge equilibrium and the same authors proposed a condition that overcomes the difficulty in these papers.

Let $M = I$, consider $R_i = \{i\}$, for any $i \in I$. It is obvious that the family $R = \{R_i\}_{i \in I}$ is a partition of the set of players I , and let $S_i = -i$, for all $i \in I$. In this case the definition 2.1 reduces to the following definition of Berge equilibrium in the sense of Zhukovskii [1994].

Definition 2.2 *A feasible strategy $\bar{x} \in X$ is a Berge equilibrium in the sense of Zhukovskii for game (2.1) if*

$$f_i(\bar{x}) \geq f_i(x_{-i}, \bar{x}_i),$$

for each given $i \in I$ and $x_{-i} \in X_{-i}$.

This definition means that when a player $i \in I$ plays his strategy \bar{x}_i from the Berge equilibrium \bar{x} , he cannot obtain a maximum payoff unless the remaining players $-i$ willingly (or obliged) play the strategy \bar{x}_{-i} from the Berge equilibrium \bar{x} . In other words, if at least one of the players of coalition $-i$ deviates from his equilibrium strategy, the payoff of the player i in the resulting strategy profile would be at most equal to his payoff $f_i(\bar{x})$ in Berge equilibrium. This equilibrium can be used as an alternative solution when there is no Nash equilibrium or when there are many in a game.

2.1 Strong Berge Equilibrium

Berge [1957] introduced the strong Berge equilibrium which is a refinement of a Nash equilibrium as shown by Larbani and Nessah [2001]. In this section, we recall the definition of strong Berge equilibrium and its properties.

Definition 2.3 *(Larbani and Nessah [2001]) A strategy profile $\bar{x} \in X$ is said to be strong Berge equilibrium (SBE) of game (2.1), if*

$$\forall i \in I, \forall j \in -i, f_j(\bar{x}_i, y_{-i}) \leq f_j(\bar{x}), \forall y_{-i} \in X_{-i}. \quad (2.2)$$

If a player i chooses his strategy \bar{x}_i of a \bar{x} which is a SBE, then the coalition $-i$ cannot improve the earnings of all its players, *i.e.* by deviating from \bar{x} . In other words, SBE is stable against deviations of any coalition of type $-i, i \in I$.

1. The SBE is very stable. Indeed, if a player $i \in I$ chooses his (her) strategy \bar{x}_i in a SBE, then he obliges all the players in the coalition $-i$ to choose their strategies in the same SBE; if any player j in $-i$ deviates from his strategy \bar{x}_j , he will not be better off.

2. SBE is also a Nash equilibrium. Indeed, let $\bar{x} \in X$ be a SBE of game (2.1), and let $i \in I$, suppose that, player i choose a strategy x_i , then for all $j \in -i$, we have $f_i(x_i, \bar{x}_{-i}) = f_i(\bar{x}_j, \bar{x}_{-\{i,j\}}, x_i) \leq f_i(\bar{x})$. Since i is arbitrarily chosen in I , then \bar{x} is a Nash equilibrium.
3. SBE is individually rational, *i.e.*, the SBE gives
any player at least the minimum profit that he can guaranteed against all possible behaviors of the coalition $-i$.
4. In general, the SBE is not Pareto optimal, *i.e.*, the SBE does not satisfy the collective rationality principle.
5. If $n = 2$, then the concepts of strong Berge equilibrium and Nash equilibrium are identical.

The following Lemmas Larbani and Nessah [2001] show that SBE has exactly the same characteristics Nash equilibrium has in two-person zero-sum games, namely, the interchangeability and the equivalence.

Lemma 2.1 *Let the game (2.1) be a zero-sum, i.e.,*

$$\sum_{i=1}^n f_i(x) = 0, \forall x \in X.$$

If \bar{x} and \tilde{x} are two different SBE for the game (2.1), then $\forall i \in I, f_i(\bar{x}) = f_i(\tilde{x})$.

Lemma 2.2 *Assume that*

$$\sum_{i=1}^n f_i(x) = 0, \forall x \in X.$$

Let $x^l, l = 1, \dots, s$ and $s = 1, \dots, n$ be s different SBEs of game (2.1) and let $\{K_l\}_{l=1, \dots, s}$ be a partition of the set players I such that for any $i \in K_l$, the player i chooses his (her) strategy in SBE x^l , then $\tilde{x} = (x_{K_1}, x_{K_2}, \dots, x_{K_s})$ is also an SBE of game (2.1).

Lemma 2.1 shows that in a game with a zero-sum, the SBE has the equivalence property, *i.e.*, the payoff functions have the same value for all SBE's. In this case, contrary to the Nash equilibrium, for the players, the problem of selection of SBE where it will choose their strategy, does not arise. Lemma 2.2 shows that in a zero-sum game, the SBE has the interchangeability property, *i.e.*, if the players choose their strategies in various SBEs, the obtained strategy profile is also an SBE. Let us note that this property is not true for Nash equilibrium in games involving $n > 2$ players. Thus, we can conclude that SBE is a "solution" for an n -person non cooperative zero-sum game, as Nash equilibrium is a "solution" for two-person Zero-sum games.

Theorem 2.1 (Larbani and Nessah [2001]) Assume that

1. the strategy sets $X_i, i \in I$ are nonempty compact and convex subsets of locally convex Hausdorff spaces,
2. $\forall i \in I$, the payoff function $f_i(x)$ is continuous over X and the function $y_{-i} \mapsto f_j(x_i, y_{-i})$ is quasi-concave over X_{-i} , for all $j \neq i \in I$ and $\forall x_i \in X_i$;
3. $\forall x \in X, \exists u \in X$, such that

$$f_j(x_i, y_{-i}) \leq f_j(x_i, u_{-i}), \forall i \in I, \forall j \in -i, \forall y_{-i} \in X_{-i}.$$

Then the game (2.1) has at least one strong Berge equilibrium.

2.2 Strong Nash Equilibrium

Aumann [1959] introduced the strong Nash equilibrium which is a refinement of the Nash equilibrium and which is also a Pareto efficient. In this section, we recall the definition of strong Nash equilibrium and its properties.

Definition 2.4 A strategy profile $\bar{x} \in X$ is said to be strong Nash equilibrium (SNE) of game G (2.1), if $\forall S \in \mathfrak{S}, \forall y_S \in X_S$, the following system:

$$f_j(y_S, \bar{x}_{-S}) \geq f_j(\bar{x}), j \in S \quad (2.3)$$

with at least one strict inequality is impossible.

A strategy profile is a strong Nash equilibrium if no coalition (including the grand coalition, *i.e.*, all the players collectively) can profitably deviate from the prescribed profile. The definition immediately implies that any strong Nash equilibrium is both Pareto efficient and a Nash equilibrium. Indeed, if a coalition S deviates from its strategy \bar{x}_S some strong Nash equilibrium \bar{x} , then she cannot improve the earning of all his(her) players at the same time if the rest of the players maintains its strategy \bar{x}_{-S} of the \bar{x} . This equilibrium is stable with regard to the deviation of a coalition.

We deduce the following properties:

1. SNE is a Nash equilibrium, it suffices to consider $S = \{i\}$ in Definition 2.4.
2. SNE is an optimum of Pareto, it suffices to consider $S = I$ in Definition 2.4.
4. SNE is an element in α -core.
5. SNE is an element in β -core.

6. SNE is also an k -equilibrium, $\forall k \in \{1, 2, \dots, n\}$.

Proposition 2.1 *If $n = 2$, then the concepts of strong Berge-Pareto equilibrium and strong Nash equilibrium are identical.*

The following lemma characterizes the strong Nash equilibrium of the game (2.1).

Lemma 2.3 *A strategy profile $\bar{x} \in X$ is a strong Nash equilibrium of the game (2.1) if and only if, for each $S \in \mathfrak{S}$, the strategy $\bar{x}_S \in X_S$ is a pareto efficient of the following sub-game $\langle X_S, f_j(\cdot, \bar{x}_{-S})_{j \in S} \rangle$.*

Proof. It is a straightforward consequence of Definition 2.4. ■

3 Existence Results

Let us recall the following definition of Pareto efficient.

Definition 3.1 (Moulin [1979]) *A strategy profile $\bar{x} \in X$ of the game (2.1) is said to be Pareto efficient if the system $f_j(y) \geq f_j(\bar{x})$, $j \in I$ with at least one strict inequality is impossible.*

We denote by PE the set of all strategy profiles Pareto efficient.

Definition 3.2 *$\bar{x} \in X$ is said to be strong Berge and Pareto equilibrium (SBPE) of game (2.1), if \bar{x} is a strong Berge equilibrium which is also Pareto efficient of the same game.*

We have the following theorem.

Theorem 3.1 *Any SBPE of the game (2.1) is also an SNE for this game.*

Proof. Let $\bar{x} \in X$ be an SBPE of the game (2.1), then by definition, we have:

$$\begin{cases} 1) \forall i \in I, \forall j \in -i, f_j(\bar{x}_i, y_{-i}) \leq f_j(\bar{x}), \forall y_{-i} \in X_{-i} \\ 2) \bar{x} \text{ is a Pareto efficient.} \end{cases} \quad (3.1)$$

Suppose that \bar{x} is not SNE, then there exists $S_0 \in \mathfrak{S}$ and $\tilde{y}_{S_0} \in X_{S_0}$ such that:

$$\begin{cases} 1) \forall h \in S_0, f_h(\tilde{y}_{S_0}, \bar{x}_{-S_0}) \geq f_h(\bar{x}), \\ 2) \exists h \in S_0, f_h(\tilde{y}_{S_0}, \bar{x}_{-S_0}) > f_h(\bar{x}). \end{cases} \quad (3.2)$$

The system (3.2) implies that

$$\sum_{h \in S_0} f_h(\tilde{y}_{S_0}, \bar{x}_{-S_0}) > \sum_{h \in S_0} f_h(\bar{x}). \quad (3.3)$$

Case 1. If $S_0 = I$. Inequality (3.2), implies that \bar{x} is not Pareto efficient for the game (2.1). This contradicts assumption 2) of system (3.1).

Case 2. If $S_0 \neq I$, then $-S_0 \neq \emptyset$. Let $i_0 \in S_0$, thus, $S_0 \subset -i_0$. Let $L = (-i_0) - S_0$, then we have by assumption 1) of the system (3.1):

$$\forall j \in -i_0 = S_0 \cup L, f_j(\bar{x}_{i_0}, y_{-i_0}) \leq f_j(\bar{x}), \forall y_{-i_0} \in X_{-i_0}.$$

Let $y_{-i_0} = (\bar{x}_L, \tilde{y}_{S_0})$, $\forall j \in S_0$ and $y_{-i_0} = \bar{x}_{-i_0}$, $\forall j \in L$ in the last inequality, then we obtain

$$\forall j \in S_0, f_j(\bar{x}_{-S_0}, \tilde{y}_{S_0}) \leq f_j(\bar{x}). \quad (3.4)$$

The system (3.4) implies that $\sum_{j \in S_0} f_j(\bar{x}_{-S_0}, \tilde{y}_{S_0}) \leq \sum_{j \in S_0} f_j(\bar{x})$. This contradicts inequality (3.3).

This completes the proof. ■

In the following theorem, we establish the existence of SBPE of game (2.1) by the Ky Fan Inequality.

Ky Fan Inequality. *Let X be a nonempty, convex and compact set in a locally convex Hausdorff space E and let f be a real valued function defined on $X \times X$. Suppose that the following conditions are satisfied*

- 1) *the function $x \mapsto f(x, y)$ is continuous over X , $\forall y \in X$ and the function $y \mapsto f(x, y)$ is lower semicontinuous over X , $\forall x \in X$,*
- 2) *the function $y \mapsto \Psi(x, y)$ is quasi-concave over X , $\forall x \in X$.*

Then, there exists $\bar{x} \in X$ such that

$$\sup_{y \in X} \Psi(\bar{x}, y) = \Psi(\bar{x}, \bar{x}) \leq \sup_{y \in X} \Psi(y, y).$$

Let

$$\Delta = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n / \lambda_i \geq 0, \forall i = 1, \dots, n \text{ and } \sum_{j \in I} \lambda_j = 1\}.$$

be the simplex of \mathbb{R}^n . Let us consider the following function.

$$\psi_\lambda : X \times (\tilde{X} \times X) \rightarrow \mathbb{R}$$

defined by $(x, (\tilde{y}, z)) \mapsto \psi_\lambda(x, (\tilde{y}, z)) = \sum_{i \in I} \sum_{j \in -i} \{f_j(x_i, y_{-i}) - f_j(x)\} + \sum_{j \in I} \lambda_j \{f_j(z) - f_j(x)\}$, where $\tilde{X} = (\prod_{i \in I} \prod_{j \in -i} X_{-i}^j)$ with $X_{-i}^j = X_{-i}$, $\forall i \in I$, $\forall j \in -i$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta$.

Remark 3.1 We have:

$$\forall x \in X, \sup_{(\tilde{y}, z) \in \tilde{X} \times X} \psi_\lambda(x, (\tilde{y}, z)) \geq 0,$$

because for any $x \in X$, if we take $\tilde{y} = ((\underbrace{x_{-1}, \dots, x_{-1}}_{(n-1) \text{ times}}, \underbrace{x_{-2}, \dots, x_{-2}}_{(n-1) \text{ times}}, \dots, \underbrace{x_{-n}, \dots, x_{-n}}_{(n-1) \text{ times}}))$ and $z = x$, we obtain $\psi_\lambda(x, (\tilde{y}, z)) = 0$.

We have the following theorem.

Theorem 3.2 Assume that:

1. the strategy sets $X_i, i \in I$, are nonempty compact and convex subsets of locally convex Hausdorff spaces,
2. $\forall i \in I$, the function $f_i(x)$ is continuous and quasi-concave over X ;
3. $\exists \lambda \in \Delta$ with $\lambda_i > 0, \forall i = 1, \dots, n$, and $\forall x \in X, \exists u \in X$, such that

$$f_j(x_i, y_{-i}) \leq f_j(x_i, u_{-i}), \forall i \in I, \forall j \in -i, \forall y_{-i} \in X_{-i}$$

$$\text{and } \sum_{h \in I} \lambda_h f_h(z) \leq \sum_{h \in I} \lambda_h f_h(u), \forall z \in X.$$

Then the game (2.1) has at least one SBPE which is also an SNE.

Proof. Let us consider the following function:

$$\Omega_\lambda : X \times X \rightarrow \mathbb{R}$$

defined by $(x, y) \mapsto \Omega_\lambda(x, y) = \sum_{i \in I} \sum_{j \in -i} \{f_j(x_i, y_{-i}) - f_j(x)\} + \sum_{h \in I} \lambda_h \{f_h(y) - f_h(x)\}$.

From assumption 3) of Theorem 3.2, we deduce that $\forall x \in X, \exists u \in X$, such that

$$\psi_\lambda(x, (\tilde{y}, z)) \leq \Omega_\lambda(x, u), \forall (\tilde{y}, z) \in \tilde{X} \times X. \quad (3.5)$$

From assumptions 1)-2) of Theorem 3.2, we deduce that $x \mapsto \Omega_\lambda(x, u)$ is continuous over $X, \forall u \in X$ and the function $u \mapsto \Omega_\lambda(x, u)$ is quasi-concave over $X, \forall x \in X$. Since Ω_λ is defined on the compact and convex X , then all conditions of Ky Fan Inequality Theorem are satisfied. Consequently, $\exists \bar{x} \in X$ such that $\sup_{u \in X} \Omega_\lambda(\bar{x}, u) \leq \sup_{u \in X} \Omega_\lambda(u, u)$.

By construction of Ω_λ , we have $\Omega_\lambda(u, u) = 0, \forall u \in X$. Therefore,

$$\sup_{u \in X} \Omega_\lambda(\bar{x}, u) \leq 0. \quad (3.6)$$

Inequalities (3.5) and (3.6) implies $\sup_{(\tilde{y}, z) \in \tilde{X} \times X} \psi_\lambda(\bar{x}, (\tilde{y}, z)) \leq 0$. According to Remark 3.1, we obtain:

$$\sup_{(\tilde{y}, z) \in \tilde{X} \times X} \psi_\lambda(\bar{x}, (\tilde{y}, z)) = 0.$$

Therefore,

$$\sum_{i \in I} \sum_{j \in -i} \{f_j(\bar{x}_i, y_{-i}) - f_j(\bar{x})\} + \sum_{j \in I} \lambda_j \{f_j(z) - f_j(\bar{x})\} \leq 0, \quad \forall (\tilde{y}, z) \in \tilde{X} \times X. \quad (3.7)$$

Letting $y_{-i} = \bar{x}_{-i}$, $\forall i \in I$ and $\forall j \in -i$ in (3.7), it becomes: $\sum_{j \in I} \lambda_j \{f_j(z) - f_j(\bar{x})\} \leq 0$, $\forall z \in X$ which implies that \bar{x} is a Pareto efficient strategy profile of the game (2.1).

Letting $z = \bar{x}$, in (3.7), it becomes:

$$\sum_{i \in I} \sum_{j \in -i} \{f_j(\bar{x}_i, y_{-i}) - f_j(\bar{x})\} \leq 0, \quad \forall \tilde{y} \in \tilde{X}. \quad (3.8)$$

Now let us prove that:

$$\forall y_{-i} \in X_{-i}, \quad f_j(\bar{x}_i, y_{-i}) \leq f_j(\bar{x}), \quad \forall i \in I, \quad \forall j \in -i.$$

i.e. \bar{x} is an SBE. Let i_0 and j_0 be two elements of I such that $j_0 \in -i_0$.

For $i \in -i_0$, $j \in -i$, let $y_{-i}^j = \bar{x}_{-i}$ and for $i = i_0$, $j \in -\{j_0, i_0\}$, $y_{-i}^j = \bar{x}_{-i}$, then (3.8) becomes

$$\forall y_{I-i_0}^{j_0} \in X_{I-i_0}, \quad f_{j_0}(\bar{x}_{i_0}, y_{I-i_0}^{j_0}) \leq f_{j_0}(\bar{x}).$$

Since i_0 and j_0 are arbitrarily chosen in I . Then,

$$\forall y_{-i} \in X_{-i}, \quad f_j(\bar{x}_i, y_{-i}) \leq f_j(\bar{x}), \quad \forall i \in I, \quad \forall j \in -i.$$

We conclude that \bar{x} is an SBPE of the game (2.1) and, by Theorem 3.1, \bar{x} is an SNE of the game (2.1). ■

Let us consider the following function:

$$\Gamma : X \times \tilde{X} \rightarrow \mathbb{R}$$

defined as $(x, \tilde{y}) \mapsto \Gamma(x, \tilde{y}) = \sum_{i \in I} \sum_{j \in -i} \{f_j(x_i, y_{-i}) - f_j(x)\}$.

Then, $\forall (x, \tilde{y}) \in X \times \tilde{X}$, we have $\Gamma(x, \tilde{y}) = \psi_\lambda(x, (\tilde{y}, x))$, $\forall \lambda \in \Delta$.

Taking into account to Remark 3.1 and Theorem 3.2, we deduce the following proposition.

Proposition 3.1 Suppose that in the game (2.1) f_i is continuous on X and X_i is compact, for $i \in I$.

Let PE be the set of Pareto efficient strategy profiles of the game (2.1) and define

$$\alpha = \min_{x \in PE} \max_{\tilde{y} \in \tilde{X}} \Gamma(x, \tilde{y}). \quad (3.9)$$

Then the following propositions are equivalent:

1. game (2.1) has at least one SBPE.
2. $\alpha = 0$.

Note that α exists because of compactness and continuity assumptions in Proposition 3.1.

Remark 3.2 If all conditions of Proposition 3.1 are satisfied, then the set PE is nonempty for the game (2.1), i.e., $PE \neq \emptyset$. Indeed, the functions f_i , $i \in I$, are continuous over the compact $X = \prod_{i \in I} X_i$, then by the Weierstrass theorem, there exists $\bar{x} \in X$ such that

$$\max_{x \in X} \sum_{i=1}^n f_i(x) = \sum_{i=1}^n f_i(\bar{x}).$$

Which implies that \bar{x} is Pareto efficient for the game (2.1).

Remark 3.3 From Theorem 3.1 and Proposition 3.1, we deduce that if $\alpha = 0$, then the game (2.1) has at least one SNE.

From Proposition 3.1, we deduce the following method for the computation of SBPE of the game (2.1).

Suppose that all conditions of the Proposition 3.1 are satisfied.

Step 0. Compute the set of Pareto efficient PE of game (2.1).

Step 1. Calculate the value α in (3.9).

Step 2.

If $\alpha > 0$, then game (2.1) has no SBPE.

If $\alpha = 0$, then a strategy profile $\bar{x} \in X$ verifying $\max_{\tilde{y} \in \tilde{X}} \Gamma(\bar{x}, \tilde{y}) = 0$ are SBPE of game (2.1).

Figure 1: Procedure for the determination of a SBPE

In the above procedure for computation of an SBPE, it is necessary to compute the set PE , which may be difficult. In the following, we establish another procedure that does not require the knowledge of the set PE . For this purpose we will use the notion weakly Pareto efficient strategy profile.

Definition 3.3 (Moulin [1979]) A strategy profile $\bar{x} \in X$ is said to be a weakly Pareto efficient of game (2.1) if the following system $f_j(y) > f_j(\bar{x})$, $j \in I$ is impossible. We denote by WPE the set of all strategy profile weakly Pareto efficient.

Note that it is well known that $PE \subset WPE$. We have the following Lemmas.

Lemma 3.1 (Moulin [1979]) If the set X_i is convex and function f_i is strictly quasi-concave, $\forall i \in I$, then $PE = WPE$.

Lemma 3.2 (Moulin [1979]) Assume that $\forall i \in I$, X_i is a nonempty, convex and compact subset of a Hausdorff locally convex space, the functions f_i , $i \in I$ are continuous and strictly quasi-concave on X .

Then $\bar{x} \in X$ is a weakly Pareto efficient startegy profile of the game (2.1) if and only if $\exists \lambda \in \Delta$ such that $\sup_{y \in X} \sum_{i \in I} \lambda_i f_i(y) = \sum_{i \in I} \lambda_i f_i(\bar{x})$.

Let us consider the following function.

$$\Upsilon : X \times \Delta \times (\tilde{X} \times X) \rightarrow \mathbb{R}$$

defined by $\Upsilon(x, \lambda, (\tilde{y}, z)) = \psi_\lambda(x, (\tilde{y}, z))$.

Then, we deduce the following theorem.

Theorem 3.3 Assume that $\forall i \in I$, X_i is a nonempty, convex and compact subset of a locally convex Hausdorff space, the functions f_i , $i \in I$ are continuous and strictly quasi-concave on X . Let

$$\beta = \min_{(x, \lambda) \in X \times \Delta} \max_{(\tilde{y}, z) \in \tilde{X} \times X} \Upsilon(x, \lambda, (\tilde{y}, z)). \quad (3.10)$$

Then, the game (2.1) has at least one SBPE if and only if $\beta = 0$.

Proof. *Sufficient Condition:* Suppose that $\beta = 0$, since the functions $x \mapsto \Upsilon(x, \lambda, (\tilde{y}, z))$ and $\lambda \mapsto \Upsilon(x, \lambda, (\tilde{y}, z))$ are continuous over the compacts X and Δ , respectively. Then Weierstrass Theorem implies that there exist $\bar{x} \in X$ and $\bar{\lambda} \in \Delta$ such that $\beta = \max_{(\tilde{y}, z) \in \tilde{X} \times X} \Upsilon(\bar{x}, \bar{\lambda}, (\tilde{y}, z)) = 0$, this equality implies $\forall \tilde{y} \in \tilde{X}$, $\forall z \in X$,

$$\Upsilon(\bar{x}, \bar{\lambda}, (\tilde{y}, z)) = \sum_{i \in I} \sum_{j \in -i} \{f_j(\bar{x}_i, y_{-i}) - f_j(\bar{x})\} + \sum_{j \in I} \bar{\lambda}_j \{f_j(z) - f_j(\bar{x})\} \leq 0. \quad (3.11)$$

Letting $z = \bar{x}$ in (3.11), we obtain then

$$\sum_{i \in I} \sum_{j \in -i} \{f_j(\bar{x}_i, y_{-i}) - f_j(\bar{x})\} \leq 0.$$

Thus \bar{x} is an SBE of the game (2.1) (See the Proof of Theorem 3.2).

From the assumptions of Theorem 3.3 and Lemma 3.1, we deduce that PE=WPE. Then it is sufficient to prove that \bar{x} is a weakly Pareto efficient strategy profile of the game (2.1). Suppose the contrary is true, then there exists $z_0 \in X$ such that:

$$\forall j \in I, f_j(z_0) > f_j(\bar{x}). \quad (3.12)$$

Since $\bar{\lambda}_j \in \Delta$, the system (3.12) implies that $\sum_{j \in I} \bar{\lambda}_j f_j(z_0) > \sum_{j \in I} \bar{\lambda}_j f_j(\bar{x})$. This contradicts the inequality (3.11), if we take $z = z_0$ and $y_{-i} = \bar{x}_{-i}$, $\forall i, j$ in (3.11). This completes the first part of the proof.

Necessary Condition: Let $\bar{x} \in X$ be an SBPE of the game (2.1). Then from Lemma 3.1 and Lemma 3.2, we deduce there exists $\bar{\lambda} \in \Delta$ such that $\max_{z \in X} \sum_{i \in I} \bar{\lambda}_i \{f_i(z) - f_i(\bar{x})\} = 0$.

Since \bar{x} is an SBE, then $\sum_{i \in I} \sum_{j \in -i} \{f_j(\bar{x}_i, y_{-i}) - f_j(\bar{x})\} \leq 0, \forall \tilde{y} \in \tilde{X}$. We conclude that

$$\max_{(\tilde{y}, z) \in \tilde{X} \times X} \Upsilon(\bar{x}, \bar{\lambda}, (\tilde{y}, z)) = 0.$$

Thus, we have:

$$\beta = \min_{(x, \lambda) \in X \times \Delta} \max_{(\tilde{y}, z) \in \tilde{X} \times X} \Upsilon(x, \lambda, (\tilde{y}, z)) \leq \max_{(\tilde{y}, z) \in \tilde{X} \times X} \Upsilon(\bar{x}, \bar{\lambda}, (\tilde{y}, z)) = 0. \quad (3.13)$$

Remark 3.1 and (3.13) imply that $\beta = 0$. This completes the proof. ■

Remark 3.4 From Theorems 3.1-3.3, we deduce that if $\beta = 0$, then the game (2.1) has at least one SNE.

From Theorem 3.3, we deduce the following procedure for the computation of SBPE of the game (2.1).

From Theorems 3.2-3.3, we deduce the following proposition.

Proposition 3.2 Assume that $\forall i \in I, X_i$ is nonempty compact, the functions $f_i, i \in I$ are continuous on X and $\exists \bar{\lambda} \in \Delta$ such that $\forall i = 1, \dots, n, \bar{\lambda}_i > 0$. If

$$\beta_{\bar{\lambda}} = \min_{x \in X} \max_{(\tilde{y}, z) \in \tilde{X} \times X} \Upsilon(x, \bar{\lambda}, (\tilde{y}, z)) = 0 \quad (3.14)$$

Then, game (2.1) has at least one SBPE.

Suppose that all conditions of Theorem 3.3 are satisfied.

Step 0. Calculate the value of β in (3.10).

Step 2.

If $\beta > 0$, then the game (2.1) has no SBPE.

If $\beta = 0$, then the strategy profile $\bar{x} \in X$ verifying $\min_{\lambda \in \Delta} \max_{(\tilde{y}, z) \in \tilde{X} \times X} \Upsilon(\bar{x}, \lambda, (\tilde{y}, z)) = 0$ are SBPE of the game (2.1).

Figure 2: Procedure for the computation of an SBPE

Example 3.1 Let us consider the following example. Assume that in game (2.1) $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, $x = (x_1, x_2)$ and

$$\begin{aligned} f_1(x) &= -x_1^2 - 2x_1 + 2x_2 \\ f_2(x) &= x_1 - 2x_2^2 - x_2. \end{aligned}$$

It is clear that the functions f_i is strictly quasi-concave over X and X_i is convex for $i = 1, 2$.

In this example $\tilde{X} = X_2 \times X_1$, we put $\hat{y} = (b, a) \in X_2 \times X_1$, $z = (c, d) \in X$, $x = (u, v)$ and $\mu = (\lambda, (1 - \lambda))$, $\lambda \in [0, 1]$.

We have $\beta = \min_{(x, \mu) \in X \times \Delta} \max_{\hat{y} \in \tilde{X}} \Upsilon(x, \hat{y}) = \min_{\lambda \in [0, 1]} \min_{u, v \in [-1, 1]} \max_{a, b, c, d \in [-1, 1]} \{[f_1(a, v) - f_1(u, v)] + [f_2(u, b) - f_2(u, v)] + [\lambda(f_1(c, d) - f_1(u, v)) + (1 - \lambda)(f_2(c, d) - f_2(u, v))]\} = \min_{u, v \in [-1, 1]} \min_{\lambda \in [0, 1]} \max_{a, b, c, d \in [-1, 1]} \{[-a^2 - 2a] + [-2b^2 - b] + [-\lambda c^2 + (1 - 3\lambda)c] + [-2(1 - \lambda)d^2 + (3\lambda - 1)d] + [(1 + \lambda)u^2 + (1 + 3\lambda)u] + [2(2 - \lambda)v^2 - (2 - 3\lambda)v]\}.$

Let us consider the following function.

$$h : [0, 1] \rightarrow \mathbb{R}$$

defined by $\lambda \mapsto h(\lambda) = \min_{u, v \in [-1, 1]} \max_{a, b, c, d \in [-1, 1]} \{[-a^2 - 2a] + [-2b^2 - b] + [-\lambda c^2 + (1 - 3\lambda)c] + [-2(1 - \lambda)d^2 + (3\lambda - 1)d] + [(1 + \lambda)u^2 + (1 + 3\lambda)u] + [2(2 - \lambda)v^2 - (2 - 3\lambda)v]\}.$

Note that $\beta = \min_{\lambda \in [0, 1]} h(\lambda).$

The minimum and maximum of function Υ are reached at $\tilde{a} = \tilde{b} = \tilde{u} = 0$,

$$\tilde{c} = \begin{cases} 1, & \text{if } 0 \leq \lambda \leq 1/5 \\ \frac{1-3\lambda}{2\lambda}, & \text{if } 1/5 \leq \lambda \leq 1/3 \\ 0, & \text{if } 1/3 \leq \lambda \leq 1. \end{cases}, \quad \tilde{d} = \begin{cases} 0, & \text{if } 0 \leq \lambda \leq 1/3 \\ \frac{3\lambda-1}{4(1-\lambda)}, & \text{if } 1/3 \leq \lambda \leq 5/7 \\ 1, & \text{if } 5/7 \leq \lambda \leq 1. \end{cases}$$

$$\text{and } \tilde{v} = \begin{cases} 0, & \text{if } 0 \leq \lambda \leq 2/3 \\ \frac{3\lambda-2}{4(2-\lambda)}, & \text{if } 2/3 \leq \lambda \leq 1. \end{cases}$$

respectively. Then we obtain

$$h(\lambda) = \begin{cases} 1 - 4\lambda, & \text{if } 0 \leq \lambda \leq 1/5 \\ \frac{(3\lambda-1)^2}{4\lambda}, & \text{if } 1/5 \leq \lambda \leq 1/3 \\ \frac{(3\lambda-1)^2}{8(1-\lambda)}, & \text{if } 1/3 \leq \lambda \leq 2/3 \\ \frac{3\lambda^2+3\lambda-2}{8\lambda^2-24\lambda+16}, & \text{if } 2/3 \leq \lambda \leq 5/7 \\ \frac{-49\lambda^2+116\lambda-52}{8(2-\lambda)}, & \text{if } 5/7 \leq \lambda \leq 1. \end{cases}$$

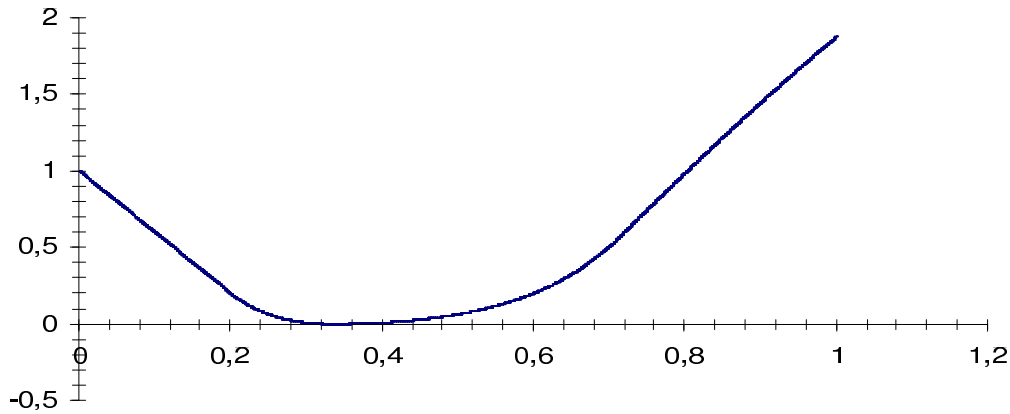


Figure 3: The graph of function h

We have $\beta = \min_{\lambda \in [0,1]} h(\lambda) = h(1/3) = 0$ (Figure 2.), then $\beta = 0$. According Theorem 3.3, the considered game has a SBPE such as $\tilde{u} = 0$ and $\tilde{v} = 0$.

4 Conclusion

In this paper, we investigated the relation between strong Berge equilibrium (SBE) and Strong Nash equilibrium (SNE). In Theorem 3.1, we showed that all strong Berge equilibrium and Pareto efficient strategy profiles (SBPE) are SNE. Based on the Ky fan inequality, we established sufficient and necessary conditions for the existence of both SBPE and SNE (Theorem 3.2-3.3). From the existence theorem we derived a procedure for computing SBPE and SNE. Extending the approach developed in this paper to other concepts of equilibrium of non cooperative game may be a worthy direction of research.

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